Big Ramsey degrees of homogeneous structures
part 2: graphs and restricted graphs

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Recall the prehistory

The prehistory

Theorem ((Infinite) Ramsey Theorem, 1930)

$$\forall_{p,k\geq 1} : \omega \longrightarrow (\omega)^{p}_{k,1}.$$

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$T(n)$ is the big Ramsey degree of n tuple in $\mathbb{Q}$.

$$T(n) = \tan^{(2n-1)}(0).$$

$$T(1) = 1, T(2) = 2, T(3) = 16, T(4) = 272,$$
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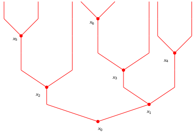
The proof (due to Laver) makes essential use of the Milliken tree theorem. This proof may seem bit arbitrary. However trees are essential (arise naturally as rich colorings). Precise bounds can be understood as a justification that this is the only approach.

Story so far

1. We well-ordered $\mathbb{Q}$ and produced tree of types



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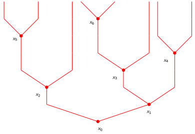
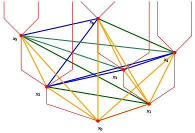


2. We a gave coloring of $\mathbb{Q}$ (by shapes of trees) so every copy of $\mathbb{Q}$ has “many colors”



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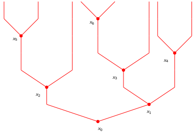

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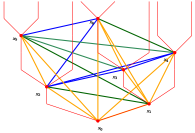

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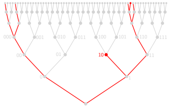

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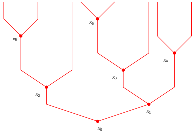

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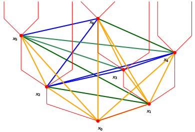

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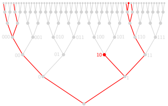
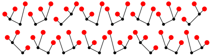


4. We described colors as structures of compatible partial orders, so "few"="many"



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- We denote by $\mathbf{R}$ the Rado (or random) graph. This is the unique homogeneous and universal countable graph. (By universal we mean that every countable graph has an embedding to $\mathbf{R}$.)
- We denote by $\mathcal{G}$ the class of all finite graphs.

Theorem

$$\forall_{\mathbf{A}\in\mathcal{G}}\exists_{T=T'(\mathbf{A})\in\omega}\forall_{k\geq 1} : \mathbf{R} \longrightarrow (\mathbf{R})^{\mathbf{A}}_{k,T}.$$

This theorem was published by Sauer in 2006 and also appears in Todorčević' Introduction to Ramsey spaces. Values of $T'(\mathbf{G})$ were characterised by Laflamme–Sauer–Vuksanović in 2010.

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A finitary version is (probably more) famous!

Theorem (Nešetřil–Rödl 1977, Abramson–Harington 1978)

$$\forall_{\mathbf{A}\in\mathcal{G}}\exists_{t=t(\mathbf{A})\in\omega}\forall_{\mathbf{B}\in\mathcal{G},k\geq 1}\exists_{\mathbf{C}} \in \mathcal{G} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k,t}.$$

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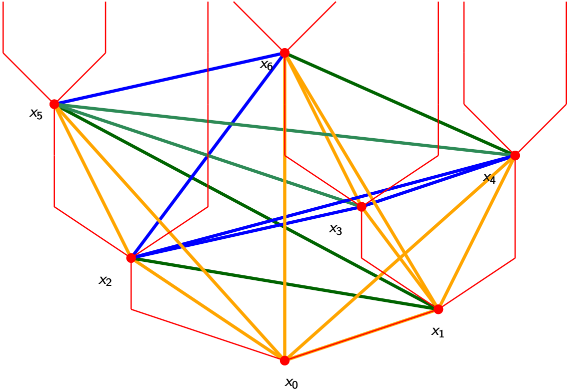

For $(\mathbb{Q}, \leq)$ we have the Sierpiński colourings. Can we do something similar for the Rado graph?

Understanding the unavoidable colourings of Rado graphs

R

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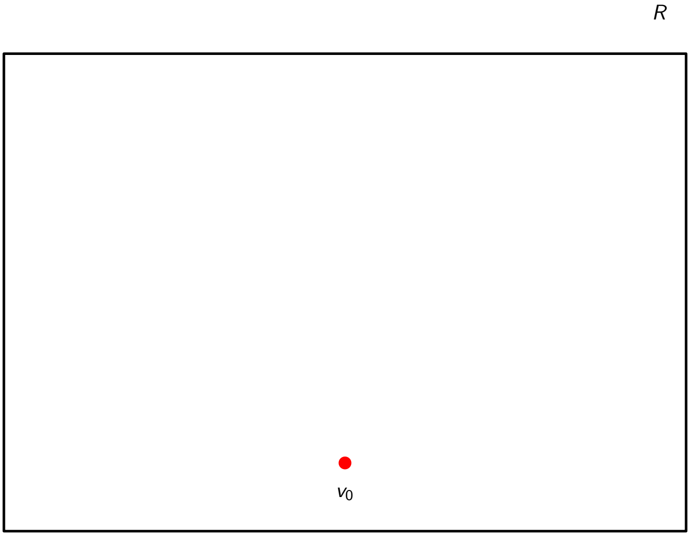

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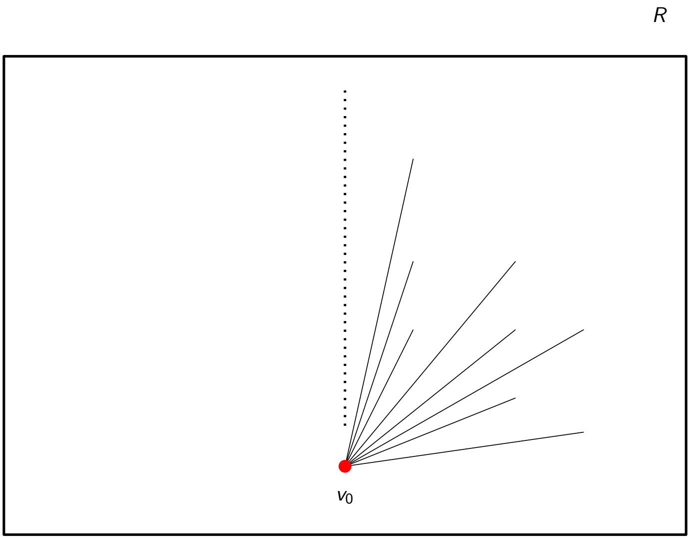

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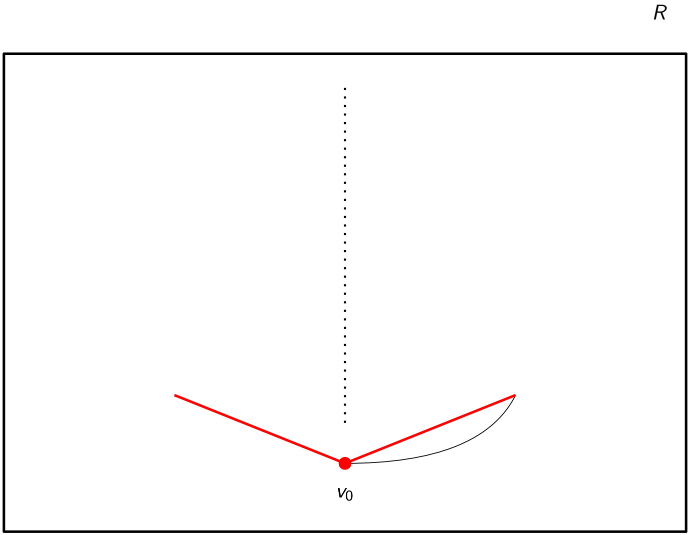

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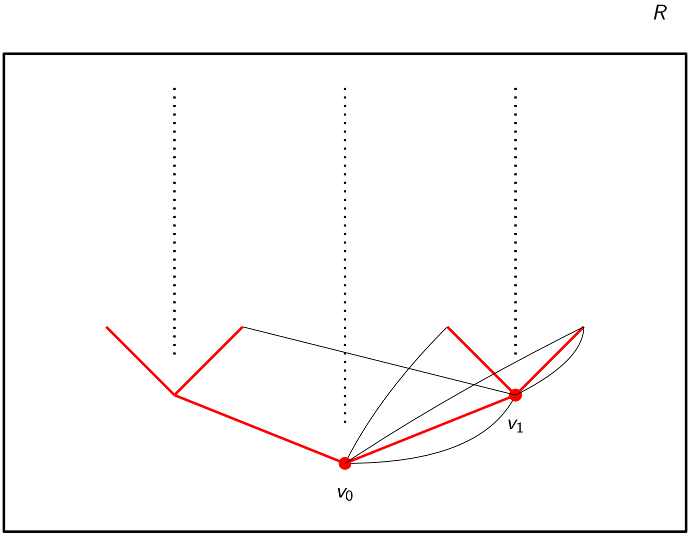

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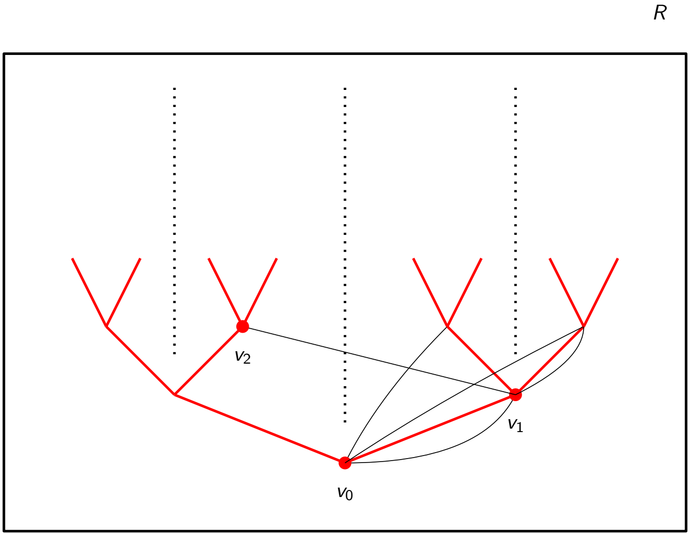

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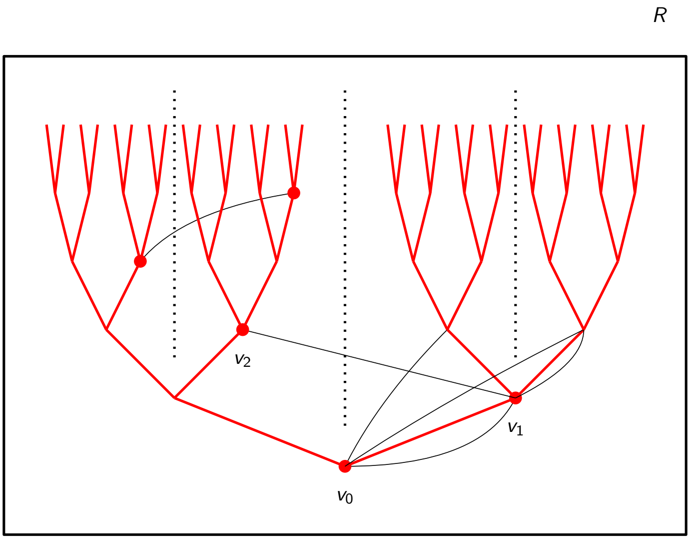

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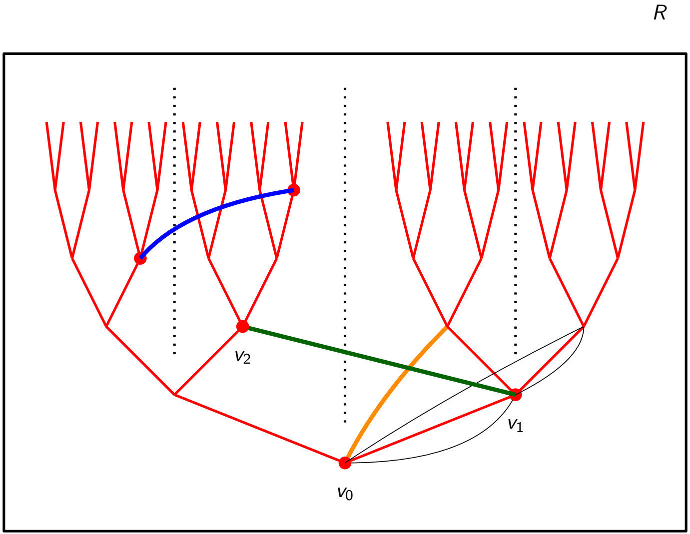
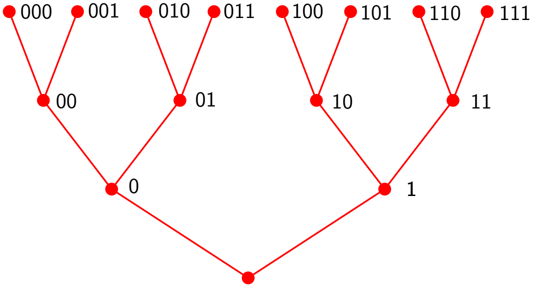


Passing number graph



Definition (Graph **G**)

We will consider graph **G**:

1. Vertices: $2^{<\omega}$
2. Vertices $a, b \in 2^{<\omega}$ satisfying $|a| < |b|$ forms and edge if and only if $b(|a|) = 1$.
3. There are no other edges.

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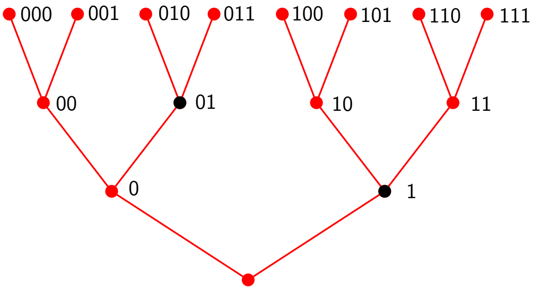

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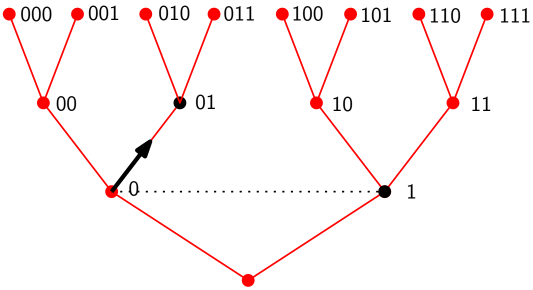

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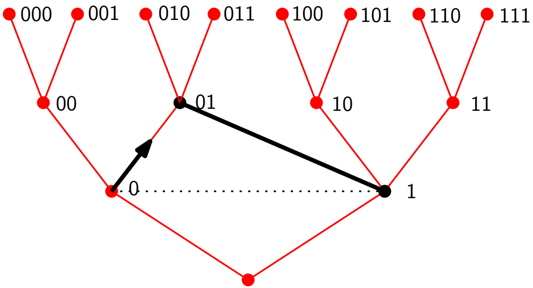

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The upper bound

Lemma

$\mathbf{G}$ *is universal: the Rado graph* $\mathbf{R}$ *embeds to* $\mathbf{G}$*.*

Proof.

Assume that the vertex set of $\mathbf{R}$ is ω. The vertex $i \in \omega$ then corresponds to a sequence a of length i with $a(j) = 1$ if and only if $i \sim j$. $\square$

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Lemma

The definition of **G** *is stable for passing into a strong subtrees: if S is a strong subtree of* $2^{<\omega}$ *then it is also a copy of* **G** *in* **G**

We thus can repeat precisely the same proof as before to obtain the upper bound on big Ramsey degrees.

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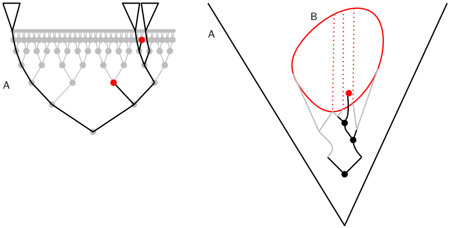
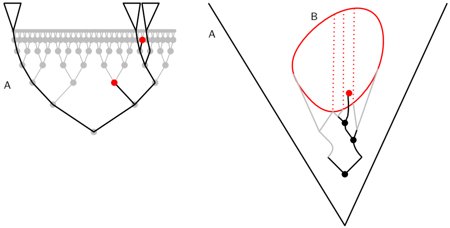
Lower bounds needs a bit more care.

Lower bound



Lower bound



"There are three ways of understanding the proof of the following Theorem 4.1. The first is to study the definition of strong diagonalization carefully and then to see that there is certainly enough room in a wide omega tree T to accommodate a strong diagonalization of T into T. The second one is to read the proof of Theorem 4.1 to the end of the construction of the function f and then to see that there is certainly enough room in a wide omega tree T to proceed with an induction argument. The third one is to read through the gory details."

N. Sauer: Coloring subgraphs of the Rado graph, Combinatorica 26 (2) (2006), 231–256. (Page 13/23).

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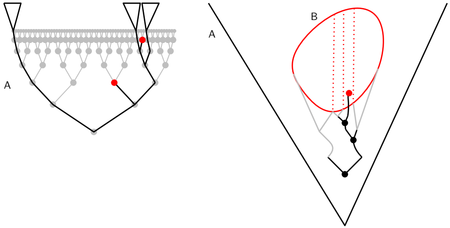

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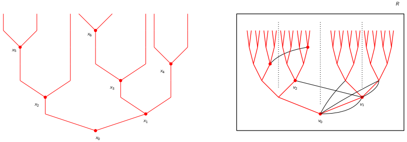
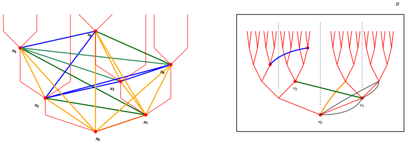
The real optimality appears later in:
Laflamme, Sauer, and Vuksanovic. Canonical partitions of universal structures. Combinatorica 26 (2) (2006): 183-206.

Big Ramsey degrees of the Rado graph

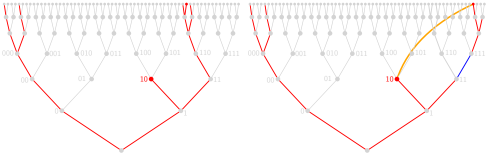
1. Enumerate R and produce tree of types



2. Give a coloring of R (by shapes of trees) so every copy of R has "many colors"



3. Apply Milliken tree theorem to find copy of R with "few colors"



4. Describe minimal set of colors as structures, so "few"="many"

Thank you for the attention

- Halpern, Läuchli: A partition theorem, Transactions of the American Mathematical Society 124 (2) (1966), 260–367.
- F. Galvin: Partition theorems for the real line, Notices Amer. Math. Soc. 15 (1968).
- F. Galvin: Errata to “Partition theorems for the real line”, Notices Amer. Math. Soc. 16 (1969).
- K. Milliken: A Ramsey theorem for trees, Journal of Combinatorial Theory, Series A 26 (3) (1979), 215–237.
- P. Erdős, A. Hajnal: Unsolved and solved problems in set theory, Proceedings of the Tarski Symposium (Berkeley, Calif., 1971). (Laver's proof is first mentioned here)
- J. Nešetřil, V. Rödl: A structural generalization of the Ramsey theorem, Bulletin of the American Mathematical Society 83 (1) (1977), 127–128.
- F. Abramson, L. Harrington: Models without indiscernibles, Journal of Symbolic Logic 43 (1978) 572–600.
- D. Devlin: Some partition theorems and ultrafilters on ω, PhD thesis, Dartmouth College, 1979.
- N. Sauer: Coloring subgraphs of the Rado graph, Combinatorica 26 (2) (2006), 231–253.
- C. Laflamme, L. Nguyen Van Thé, N. W. Sauer, Partition properties of the dense local order and a colored version of Milliken's theorem, Combinatorica 30(1) (2010), 83–104.

(See also S. Todorčević, Introduction to Ramsey spaces.)

Barbara Clatworthy (1921-2011)
Fred Payne Clatwothy (1875-1953) Autochrome, 7 x 5 inches, c1928
Mark Jacobs Collection

Some more recent results on big Ramsey degrees

1. Nguyen Van Thé (2009): Characterisation of big Ramsey degrees of homogeneous ultrametric spaces
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3. Dobrinen (2020): Big Ramsey degrees of universal homogeneous triangle-free graphs are finite
4. Dobrinen (2019+): Big Ramsey degrees of universal homogeneous K_k-free graphs are finite for every $k \geq 3$.
5. Zucker (2020+): Big Ramsey degrees of Fraïssé limits of free amalgamation classes in binary language with finitely many forbidden substructures are finite.
6. Balko, Chodounský, H., Konečný, Vena (2020+): Big Ramsey degrees of 3-uniform hypergraphs are finite.
7. J.H. (2020+): Big Ramsey degrees of partial orders and metric spaces are finite.
8. Balko, Chodounský, Dobrinen, J.H., Konečný, Nešetřil, Vena, Zucker (2021+): Big Ramsey degrees of structures described by induced cycles are finite.
9. Balko, Chodounský, Dobrinen, J.H., Konečný, Vena, Zucker (2021+): Characterisation of big Ramsey degrees of Fraïssé limits of free amalgamation classes in binary language with finitely many

Big Ramsey degrees of restricted structures

Let $\mathcal{G}_3$ be the class of all finite griangle-free grpahs.

Theorem (Dobrinen 2020)

Every (countable) universal triangle-free graph R_3 has finite big Ramsey degrees:

$$\forall_{\mathcal{A}\in\mathcal{G}_3}\exists_{T=T(|\mathbf{A}|)\in\omega}\forall_{k\geq 1} : \mathbf{R}_3 \longrightarrow (\mathbf{R}_3)^{(\mathbf{A})}_{k,T}.$$

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Let $\mathcal{P}$ be the class of all finite partial orders.

Theorem (J. H. 2020+)

Every (countable) universal partial order $(P,\leq)$ has finite big Ramsey degrees:

$$\forall_{(O,\leq)\in\mathcal{P}}\exists_{T=T(|O|)\in\omega}\forall_{k\geq 1} : (P,\leq) \longrightarrow (P,\leq)^{(O,\leq)}_{k,T}.$$

Universality: every countable partial order has embedding to $(P,\leq)$.

Tree of types of a universal triangle-free graph

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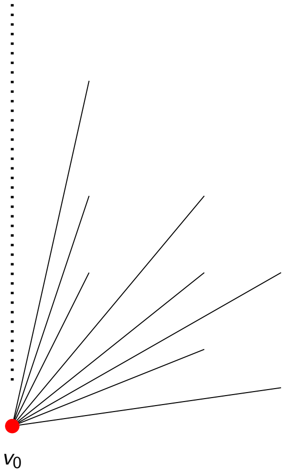

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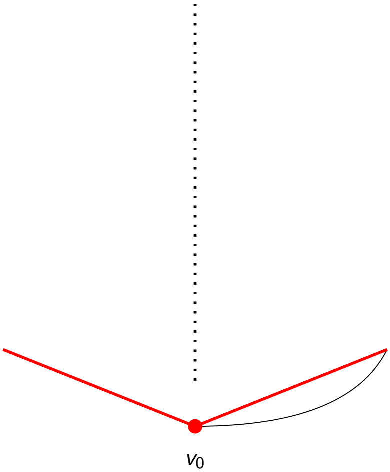

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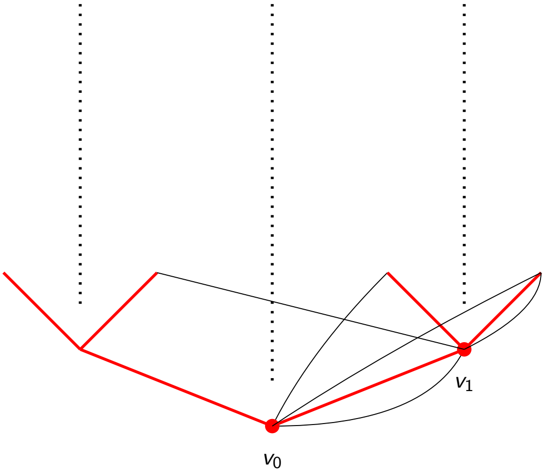

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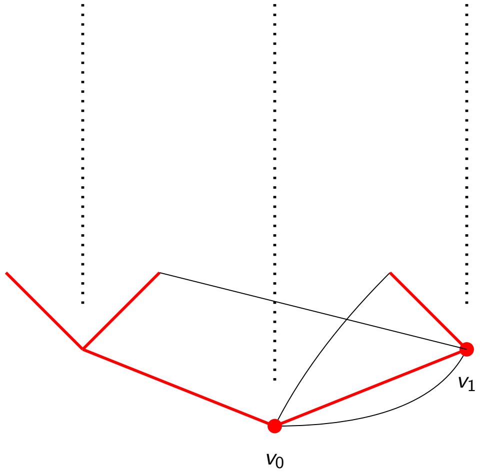

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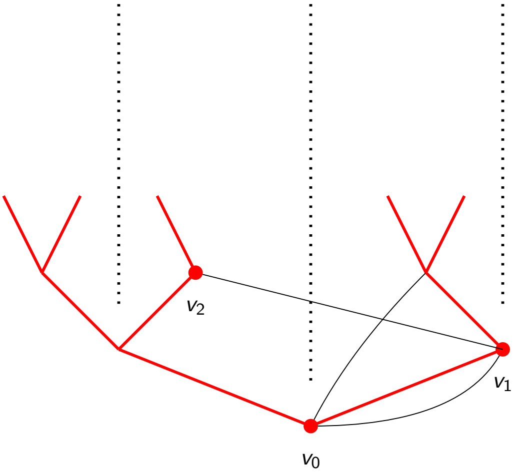

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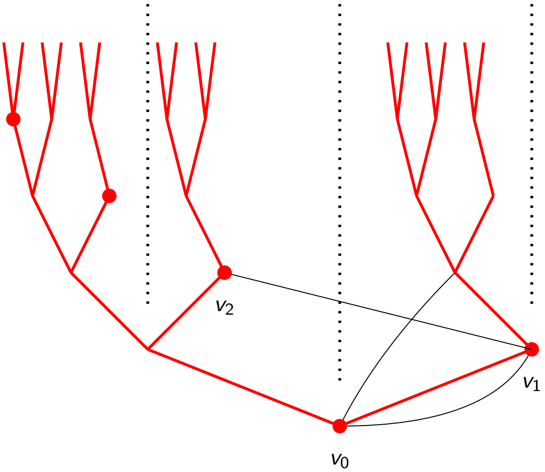

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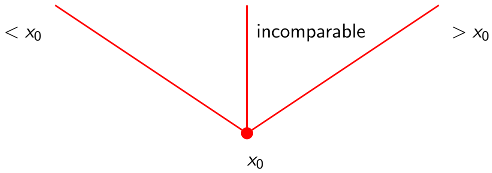

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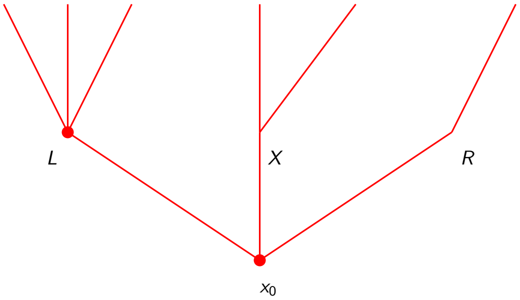

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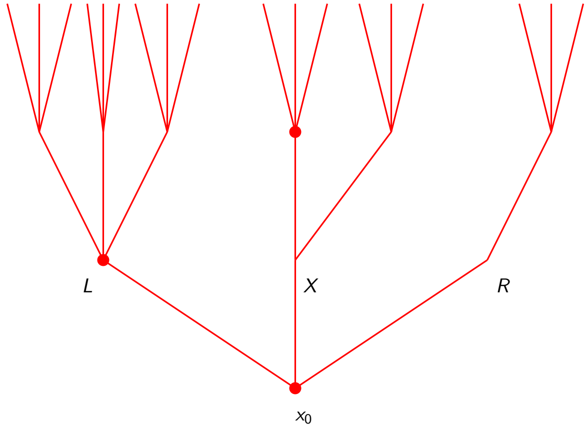

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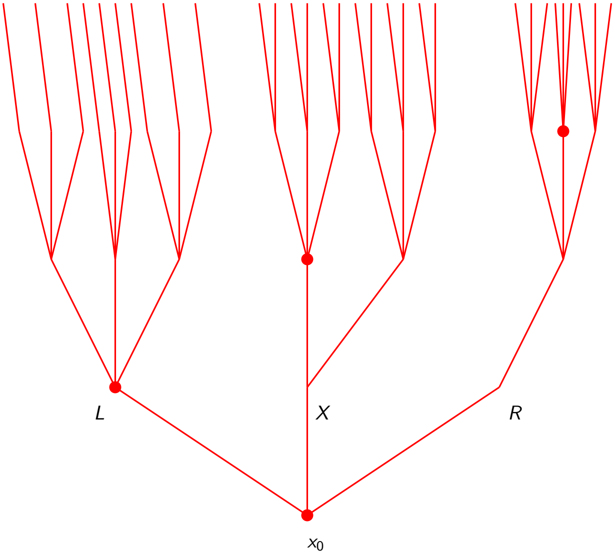

Parameter words

Definition (Parameter word)

Given a finite alphabet Σ and $k \in \omega + 1$, a *k-parameter word* is a (possibly infinite) word W in alphabet $\Sigma \cup \{\lambda_i : 0 \leq i < k\}$ such that $\forall i \in k$ word W contains λ_i and for every $j \in k - 1$, the first occurrence of λ_{j+1} appears after the first occurrence of λ_j.

Example (2-parameter word)

$\Sigma = \{\mathsf{L}, \mathsf{X}, \mathsf{R}\}$.

$$\mathsf{LRL}\lambda_0\lambda_0\mathsf{X}\lambda_1\lambda_0\mathsf{R}$$

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For set S of parameter words and a parameter word W:

$$W(S) = \{W(U) : U \in S\}.$$

Ramsey theorem for parameter words

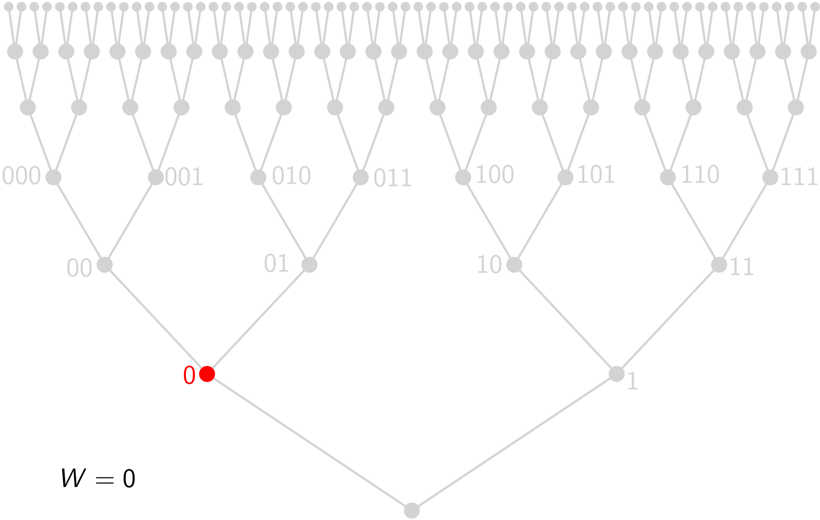
The following infinitary version of Graham–Rothschild Theorem is a direct consequence of the Carlson–Simpson theorem. It was also independently proved by Voight in 1983 (apparently unpublished):

Theorem (Ramsey theorem for parameter words)

Let Σ be a finite alphabet and $k \geq 0$ a finite integer. If the set of all finite k-parameter words in alphabet Σ is coloured by finitely many colours, then there exists a monochromatic infinite-parameter word W.

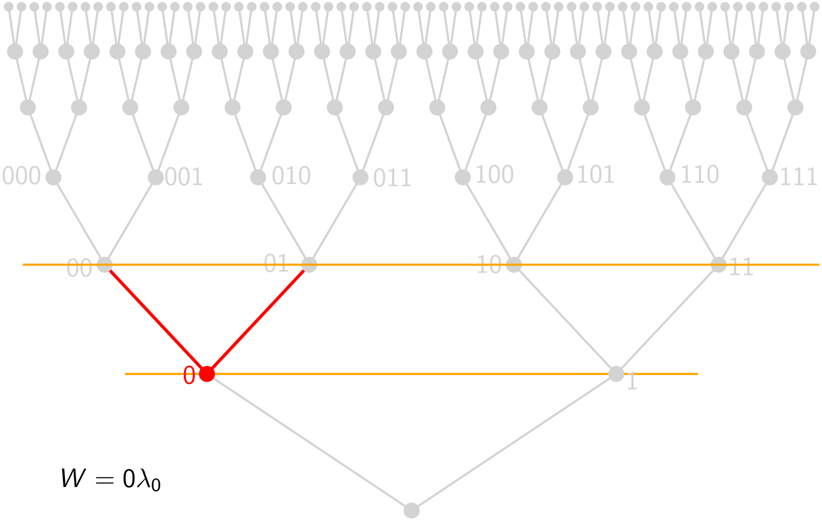
By W being monochromatic we mean that for every pair of k-parameter words U, V the colour of $W(U)$ is the same as colour of $W(V)$.

Parameter words as subtrees



$W = 0$

Parameter words as subtrees



$W = 0\lambda_0$

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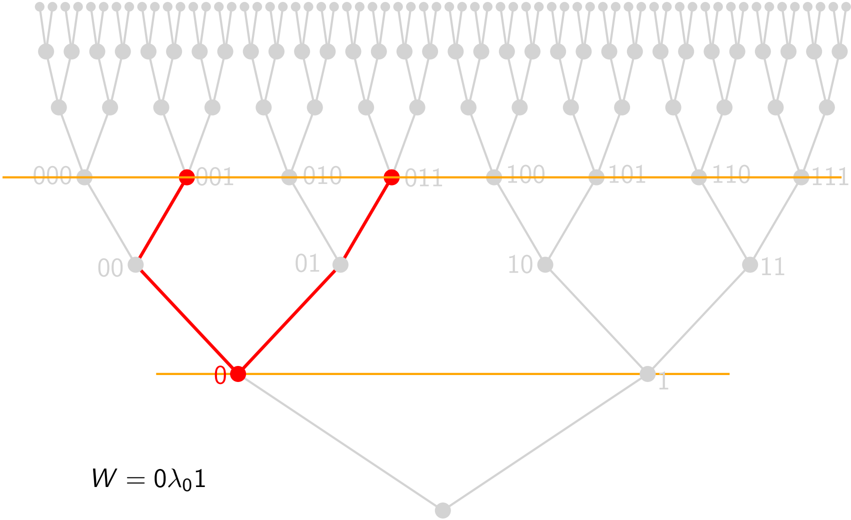

$W = 0\lambda_0 1$

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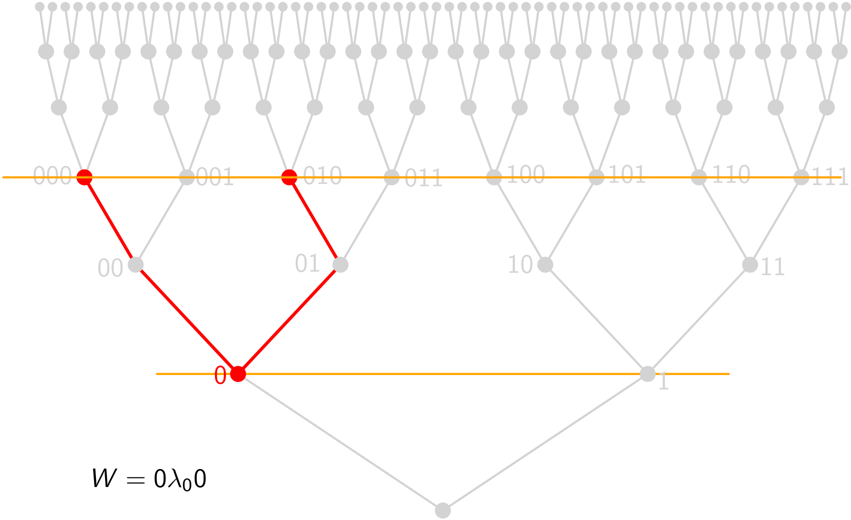

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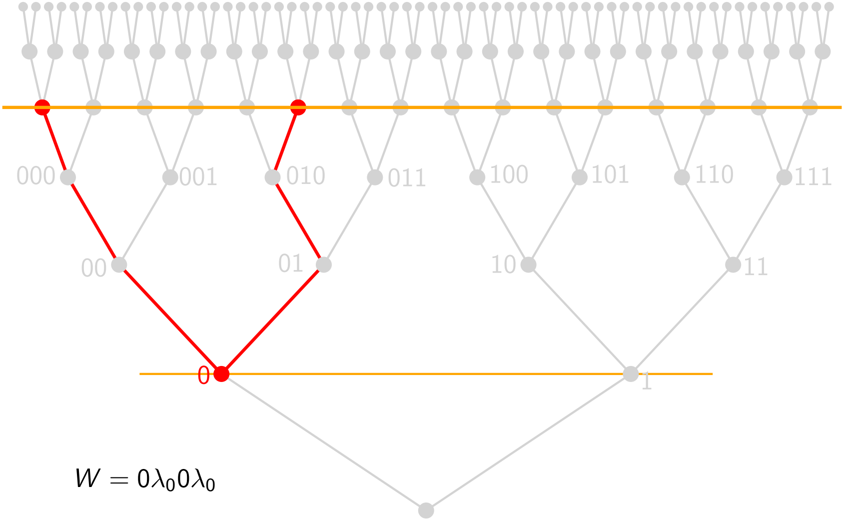

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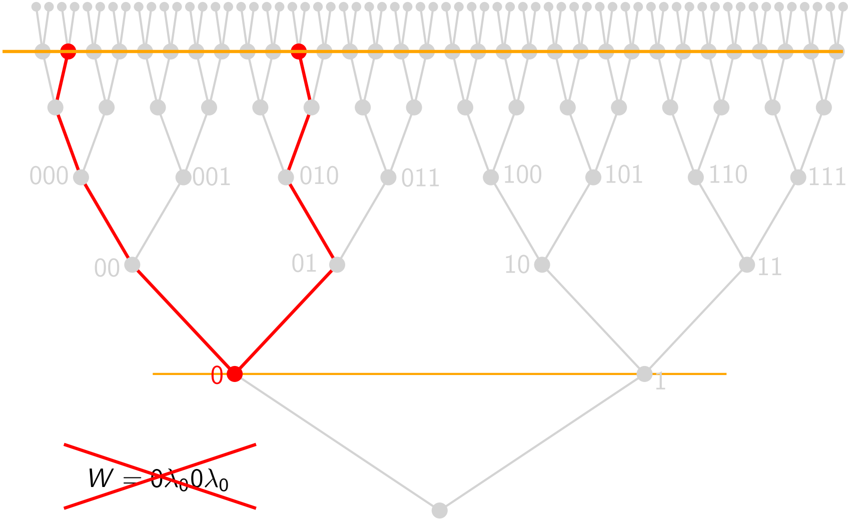

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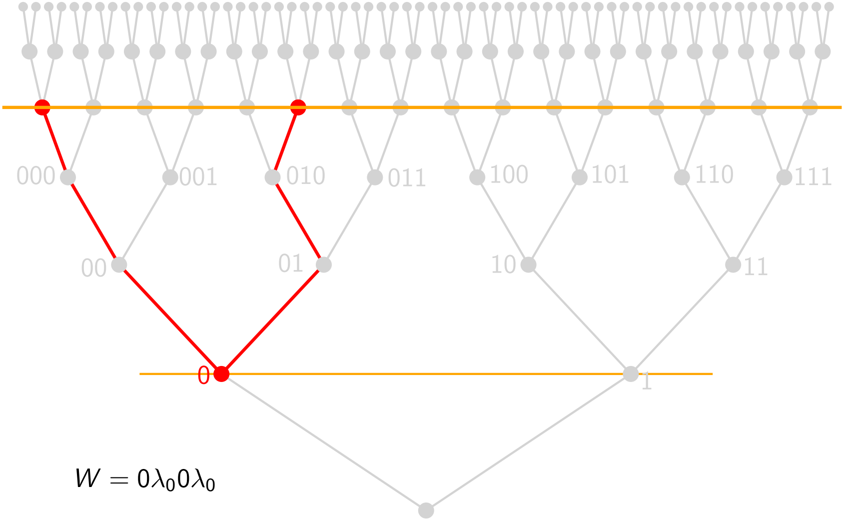

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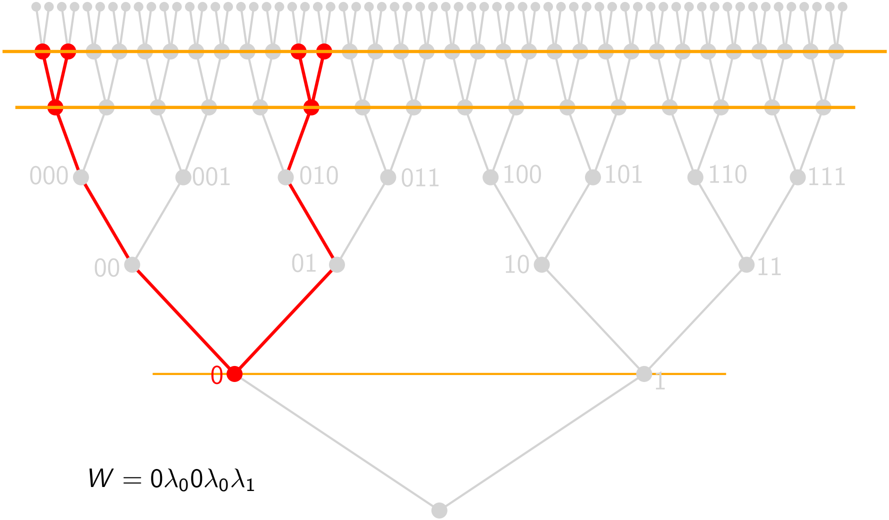

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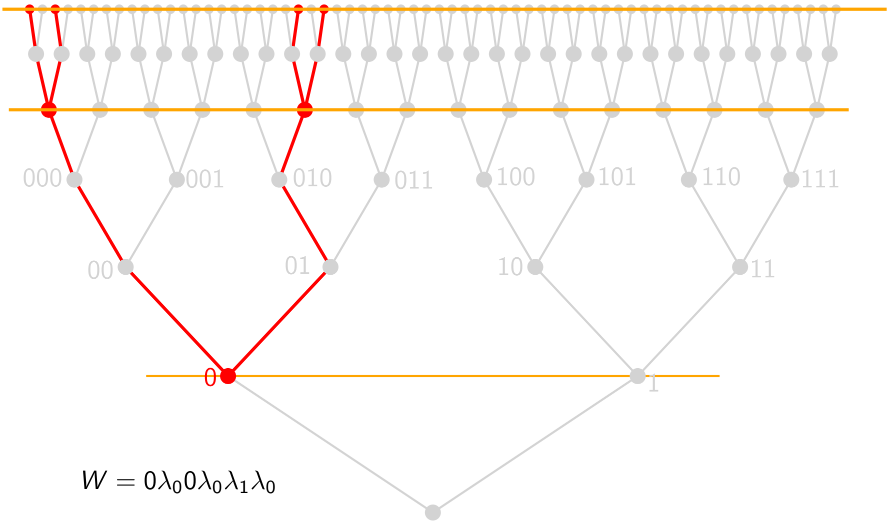

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Definition

Given a finite alphabet Σ, a finite integer $k \geq 0$ and a finite set k-parameter words, an envelope of S is every n-parameter word W (for some $n \geq k$) such that

$$\forall_{w \in S} \exists_u W(u) = w.$$

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Envelopes of $\{0, 000\}$ are: $0\lambda_0\lambda_0$, $0\lambda_0 0$, $0\lambda_0 0\lambda_1$, $\dots$.

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Let Σ be a finite alphabet, and $k, s \geq 0$ be a finite integers. Then there exists $T(|\Sigma|, s, k)$ such that for every set S of size s of k-parameter words in alphabet Σ there exists an envelope of S with at most $T(|\Sigma|, s, k)$ parameters.

Proof.

U	$=$	0	1	1	0	1	
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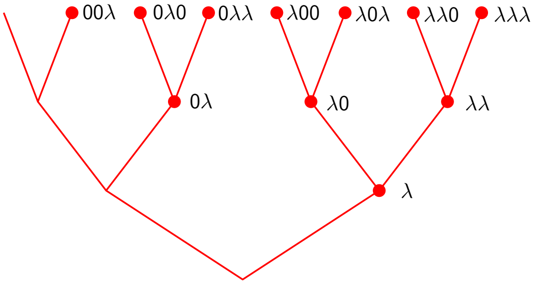
□

Triangle-free graph on 1-parameter words



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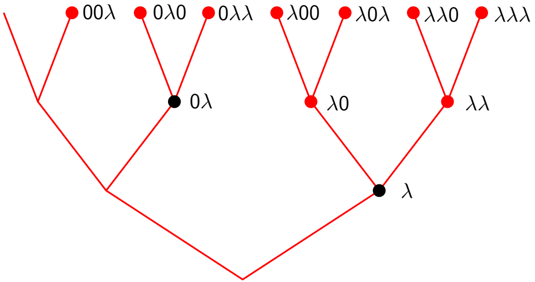

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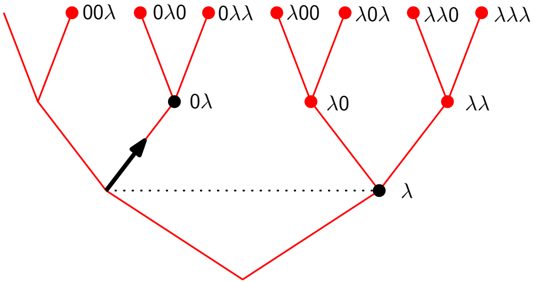

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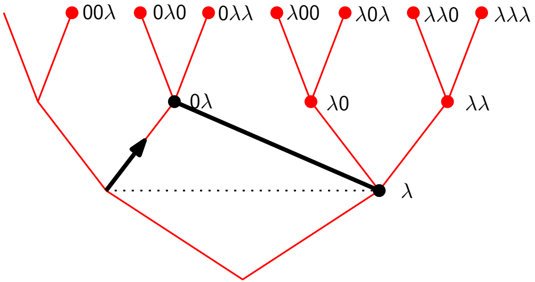

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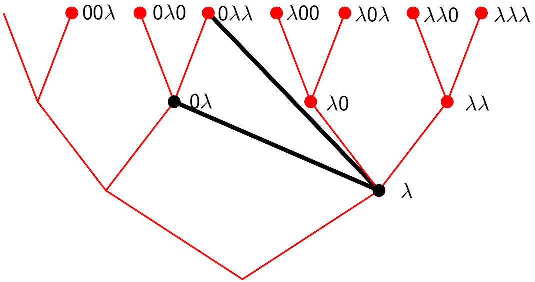

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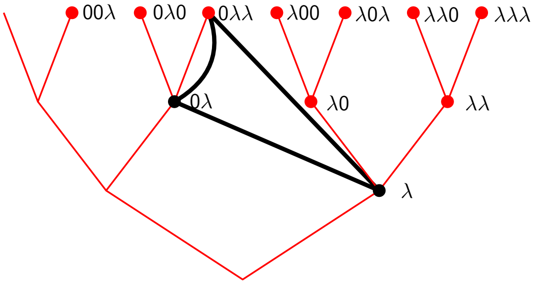

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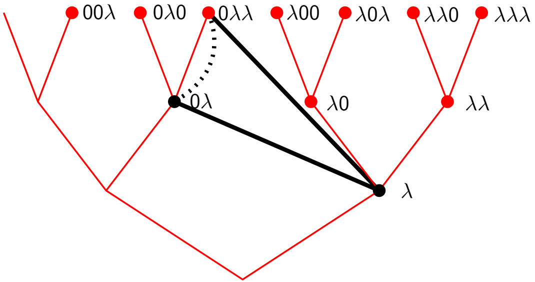

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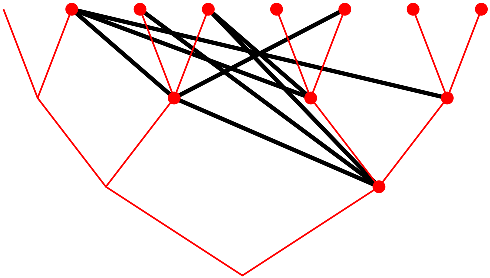

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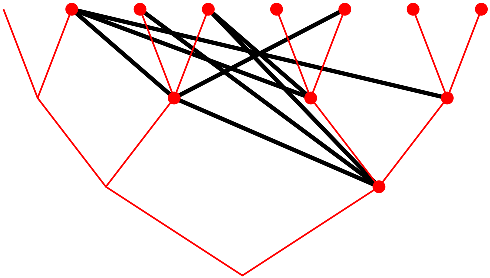

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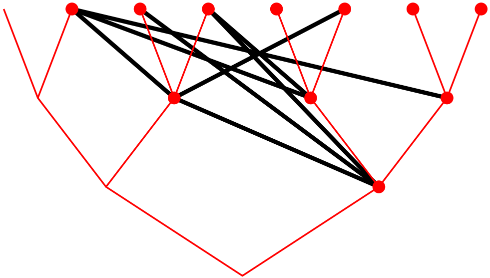

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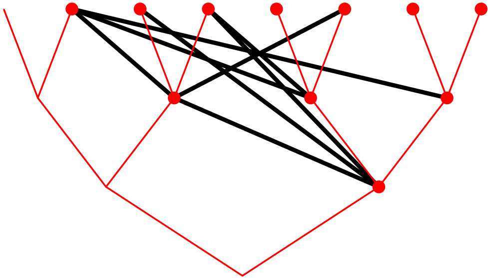
Key observation 1: G is universal triangle-free graph.
Given any triangle-free graph H with vertex set ω assign every $i \in \omega$ word w of length i putting $\forall_{j<i}\, w_j = \lambda$ iff $\{i, j\}$ is an edge of H.

Triangle-free graph on 1-parameter words



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Key observation 1: G is universal triangle-free graph.
Key observation 2: For every pair of 1-parmeter words U and V and every ω-parameter W

$$U \sim V \iff W(U) \sim W(V).$$

Observation

G is a universal triangle-free graph.

Observation

For every infinite-parameter word W it holds that $u \sim v \iff W(u) \sim W(v)$.
(Substitution is also graph embedding on $G \to G$.)

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Theorem (Dobrinen 2020)

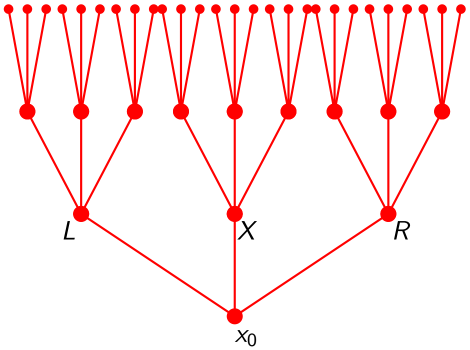
The big Ramsey degrees of universal triangle-free graph are finite.

Proof.

Fix graph A and a finite coloring of $\binom{G}{A}$. Because envelopes of copies of A are bounded, apply the theorem above for every embedding type and obtain a copy of G with bounded number of colors. $\square$


SJEZD ČESKÝCH HOSTINSKÝCH v TÁBOŘE 9. VIII 910.

Partial order on infinite ternary tree



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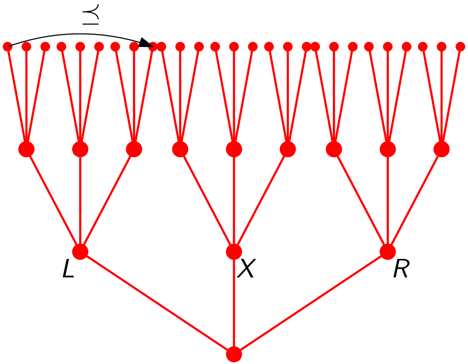

Put $\Sigma = \{\mathrm{L}, \mathrm{X}, \mathrm{R}\}$ and order $\mathrm{L} <_{\mathrm{lex}} \mathrm{X} <_{\mathrm{lex}} \mathrm{R}$.

Definition (Partial order $(\Sigma^*, \preceq)$)

For $w, w' \in \Sigma^*$ we put $w \prec w'$ if and only if there exists $0 \leq i < \min(|w|, |w'|)$ such that

1. $(w_i, w'_i) = (\mathrm{L}, \mathrm{R})$ and
2. for every $0 \leq j < i$ it holds that $w_j \leq_{\mathrm{lex}} w'_j$.

Key observations: $\preceq$ is universal partial order and is stable for substitution.


13.

More general result

Theorem (Balko, Chodounský, Hubička, Konečný, Nešetřil, Vena 2021)

Let L be a finite language consisting of unary and binary symbols, and let $\mathbf{K}$ *be a countably-infinite irreducible structure. Assume that every countable structure A has a completion to* $\mathbf{K}$ *provided that every induced cycle in* $\mathbf{A}$ *(seen as a substructure) has a completion to* $\mathbf{K}$ *and every irreducible substructure of* $\mathbf{A}$ *of size at most 2 embeds into* $\mathbf{K}$*. Then* $\mathbf{K}$ *has finite big Ramsey degrees.*

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Theorem (Balko, Chodounský, Hubička, Konečný, Nešetřil, Vena 2021)

Let L be a finite language consisting of unary and binary symbols, and let $\mathbf{K}$ be a countably-infinite irreducible structure. Assume that every countable structure A has a completion to $\mathbf{K}$ provided that every induced cycle in $\mathbf{A}$ (seen as a substructure) has a completion to $\mathbf{K}$ and every irreducible substructure of $\mathbf{A}$ of size at most 2 embeds into $\mathbf{K}$. Then $\mathbf{K}$ has finite big Ramsey degrees.

A homomorphism $f : \mathbf{A} \to \mathbf{B}$ is a mapping $f : A \to B$ such that for every $R \in L_{\mathcal{R}}$ of arity r we have: $(x_1, x_2, \ldots, x_r) \in R_{\mathbf{A}} \implies (f(x_1), f(x_2), \ldots, f(x_r)) \in R_{\mathbf{B}}$.
A homomorphism $f \colon \mathbf{A} \to \mathbf{B}$ is a homomorphism-embedding if f restricted to any irreducible substructure of A is an embedding. The homomorphism-embedding f is called a (strong) completion of $\mathbf{A}$ to $\mathbf{B}$ provided that $\mathbf{B}$ is irreducible and f is injective.

More general result

Theorem (Balko, Chodounský, Hubička, Konečný, Nešetřil, Vena 2021)

Let L be a finite language consisting of unary and binary symbols, and let **K** *be a countably-infinite irreducible structure. Assume that every countable structure A has a completion to* **K** *provided that every induced cycle in* **A** *(seen as a substructure) has a completion to* **K** *and every irreducible substructure of* **A** *of size at most 2 embeds into* **K**. *Then* **K** *has finite big Ramsey degrees.*

A homomorphism $f : \mathbf{A} \to \mathbf{B}$ is a mapping $f : A \to B$ such that for every $R \in L_{\mathcal{R}}$ of arity r we have: $(x_1, x_2, \ldots, x_r) \in R_{\mathbf{A}} \implies (f(x_1), f(x_2), \ldots, f(x_r)) \in R_{\mathbf{B}}$.
A homomorphism $f\colon \mathbf{A} \to \mathbf{B}$ is a homomorphism-embedding if f restricted to any irreducible substructure of A is an embedding. The homomorphism-embedding f is called a (strong) completion of **A** to **B** provided that **B** is irreducible and f is injective.

Corollary

The following structures have finite big Ramsey degrees:

1. *Free amalgamation structures described by forbidden triangles,*
2. *S-Urysohn space for finite distance sets S for which S-Urysohn space exists,*
3. *λ-ultrametric spaces for a finite distributive lattice λ,*
4. *Metric spaces associated to metrically homogeneous graphs of a finite diameter.*

Big picture: proof techniques

Ramsey's Theorem

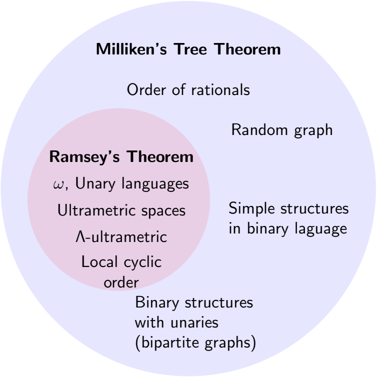
ω, Unary languages

Ultrametric spaces

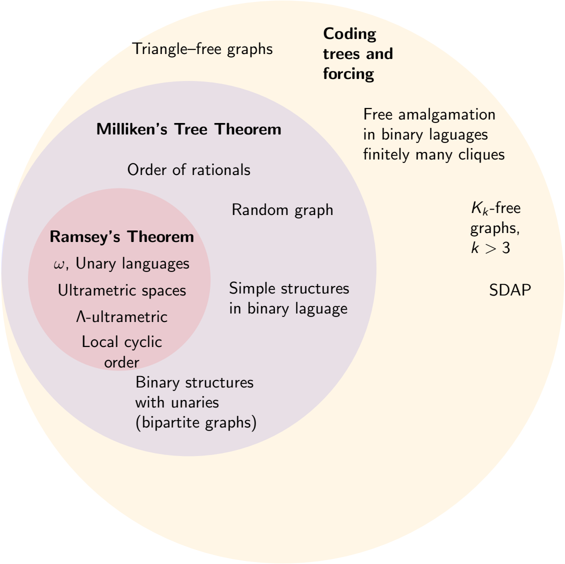
Λ-ultrametric

Local cyclic order

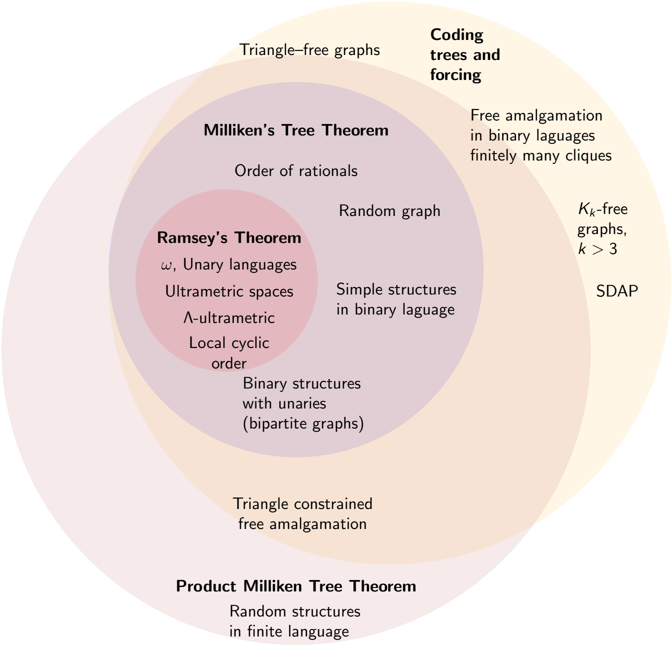
Big picture: proof techniques



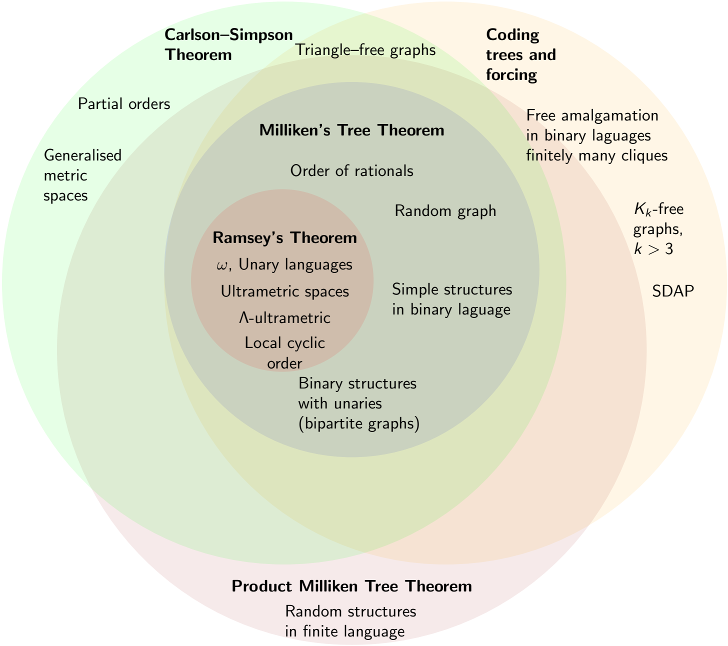
Big picture: proof techniques



Big picture: proof techniques



Big picture: proof techniques



Thank you for the attention

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